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RATIONAL RIGHT ANGLED TRIANGLES NEARLY ISOSCELES.

BY ARTEMAS MARTIN, ERIE, PA.

I. Two sides of a rational right angled triangle can not be equal, but they may differ by unity only.

Let p_n, b_n, h_n be the sides of such a right angled triangle; then we have

$$h_n^2 - b_n^2 = p_n^2,$$

and, when the hypothenuse exceeds the base by unity,

$$h_n - b_n = 1.$$

By division, $h_n + b_n = p_n^2$; therefore

$$h_n = \tfrac{1}{2}(p_n^2 + 1) \quad \text{and} \quad b_n = \tfrac{1}{2}(p_n^2 - 1).$$

It is obvious that p_n may be any odd number; therefore we may take

$$p_n = 2n + 1, \text{ then } \quad b_n = 2n^2 + 2n, \quad h_n = 2n^2 + 2n + 1,$$

which are the sides of the n^{th} triangle.

Let $n = 1$, then 3, 4, 5 are the sides of the least triangle, which we will call the *first* triangle. Take $n = 2$, then the sides of the second triangle are 5, 12, 13. The sides of the third triangle are 7, 24, 25; the sides of the fourth triangle are 9, 40, 41; &c.

The sides of the 80th triangle are 161, 12960, 12961; and the sides of the 624th triangle are 1249, 780000, 780001.

II. 1. When the *legs* differ by unity, let $\frac{1}{2}(x_n - 1) = p$, = perpendicular, $\frac{1}{2}(x_n + 1) = b$ = base and $y_n = h$ = hypothenuse.

Then $$\tfrac{1}{4}(x_n - 1)^2 + \tfrac{1}{4}(x_n + 1)^2 = y_n^2;$$

whence $$x_n^2 - 2y_n^2 = -1; \quad \ldots\ldots\ldots\ldots \quad (1)$$

or $$(x_n - y_n\sqrt{2})(x_n + y_n\sqrt{2}) = -1. \quad \ldots\ldots\ldots\ldots \quad (2)$$

Let e and f be known values of x and y; then

$$e^2 - 2f^2 = -1, \quad \ldots\ldots\ldots\ldots \quad (3)$$

and $$(e - f\sqrt{2})^{2n+1}(e + f\sqrt{2})^{2n+1} = -1. \quad \ldots\ldots\ldots \quad (4)$$

Assume $$x_n - y_n\sqrt{2} = (e - f\sqrt{2})^{2n+1},$$

then $$x_n + y_n\sqrt{2} = (e + f\sqrt{2})^{2n+1},$$

and we obtain

$$\left.\begin{array}{l} x_n = \frac{1}{2}\ \left[(e + f\sqrt{2})^{2n+1} + (e - f\sqrt{2})^{2n+1}\right] \\ y_n = \frac{1}{4}\sqrt{2}\left[(e + f\sqrt{2})^{2n+1} - (e - f\sqrt{2})^{2n+1}\right] \end{array}\right\}, \quad \ldots\ldots \quad (5)$$

where n may be 0, 1, 2, 3, 4, &c.

It is easily seen that (3) is satisfied by $e = 1, f = 1$; therefore

$$\left.\begin{array}{l} x_n = \frac{1}{2}\ \left[(1 + \sqrt{2})^{2n+1} + (1 - \sqrt{2})^{2n+1}\right] \\ y_n = \frac{1}{4}\sqrt{2}\left[(1 + \sqrt{2})^{2n+1} - (1 - \sqrt{2})^{2n+1}\right] \end{array}\right\}; \quad \ldots\ldots \quad (6)$$

and the sides of the nth triangle are

$$\left.\begin{aligned} p_n &= \tfrac{1}{2}(x_n - 1) = \tfrac{1}{4}\ \left[(1+\sqrt{2})^{2n+1} + (1-\sqrt{2})^{2n+1} - 2\right] \\ b_n &= \tfrac{1}{2}(x_n + 1) = \tfrac{1}{4}\ \left[(1+\sqrt{2})^{2n+1} + (1-\sqrt{2})^{2n+1} + 2\right] \\ h_n &= \ \ y_n \qquad\quad = \tfrac{1}{4}\sqrt{2}\left[(1+\sqrt{2})^{2n+1} - (1-\sqrt{2})^{2n+1}\right] \end{aligned}\right\}\ldots(7)$$

Expanding the values in (6) by the Binomial Theorem, we get

$$\left.\begin{aligned} x_n &= (2n+1)2^n + \left[\frac{(2n+1)(2n)(2n-1)}{1\,.\,2\,.\,3}\right]2^{n-1} \\ &\quad + \left[\frac{(2n+1)(2n)(2n-1)(2n-2)(2n-3)}{1\,.\,2\,.\,3\,.\,4\,.\,5}\right]2^{n-2} + \ldots + 1 \\ y_n &= 2^n + \left[\frac{(2n+1)(2n)}{1\,.\,2}\right]2^{n-1} \\ &\quad + \left[\frac{(2n+1)(2n)(2n-1)(2n-2)}{1\,.\,2\,.\,3\,.\,4}\right]2^{n-2} + \ldots + (2n+1) \end{aligned}\right\}\ldots(8)$$

The operation of involution is very tedious, except when n is a small number.

2. We have from (1), $x_n = \sqrt{(2y_n^2 - 1)}$. When x_n and y_n are very large numbers, the 1 under the radical may be omitted without sensible error, and we have for the superior limit of their ratio $x_n \div y_n = \sqrt{2}$; and the values of x_n and y_n are the numerators and denominators of the odd convergents to the square root of 2 expanded as a continued fraction.

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \&c.}}}$$

; the successive convergents are

$$\frac{1}{1},\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \frac{41}{29},\ \frac{99}{70},\ \frac{239}{169},\ \&c.$$

Writing $\frac{x}{y}$, $\frac{x_1}{y_1}$, $\frac{x_2}{y_2}$, $\frac{x_3}{y_3}$, $\frac{x_4}{y_4}$, &c. for the *odd*, and $\frac{v}{z}$, $\frac{v_1}{z_1}$, $\frac{v_2}{z_2}$, $\frac{v_3}{z_3}$, &c. for the ***even*** convergents, we have the relations

$$\left.\begin{aligned} x_2 &= 6x_1 - x \\ y_2 &= 6y_1 - y \end{aligned}\right\},\ \left.\begin{aligned} x_3 &= 6x_2 - x_1 \\ y_3 &= 6y_2 - y_1 \end{aligned}\right\},\ldots\ \left.\begin{aligned} x_n &= 6x_{n-1} - x_{n-2} \\ y_n &= 6y_{n-1} - y_{n-2} \end{aligned}\right\}\ldots(9)$$

Also

$$\left.\begin{aligned} z_n &= 2y_n + z_{n-1} \\ y_n &= 2z_{n-1} + y_{n-1} \\ y_{2n} &= y_n^2 + z_{n-1}^2 \end{aligned}\right\},\ \ldots\ldots\ldots\ldots\ldots\ (10)$$

which are convenient formulas for computing the sides of large triangles.

The sides of the nth triangle are

$$2y_nz_{n-1},\qquad y_n^2 - z_{n-1}^2,\qquad y_n^2 + z_{n-1}^2.$$

3. When $n = 1$, $x_1 = 7$, $y_1 = 5$, and we have 3, 4, 5, for the sides of the *first* triangle.

The sides of the second triangle are 20, 21, 29;
of the third, 119, 120, 169;
of the fourth, 696, 697, 985;
of the fifth, 4058, 4059, 5741;

. . . .

of the 39th, 433729688537240628356005653359,
433729688537240628356005653360,
613386407933224037990008001809, $= y_{39}$;
of the 40th, 2527961881478169961048032963696,
2527961881478169961048032963697,
3575077977948634627394046618865, $= y_{40}$

$z_{39} = \frac{1}{2}(y_{40} - y_{39}) = 1480845785007705294702019308528$,

$y_{80} = y_{40}^2 + z_{39}^2$, and the sides of the 80th triangle are

10588278309438211127768625972711138460195892610538807320361440
10588278309438211127768625972711138460195892610538807320361441
14974086787388384990495417211933241811765094618559069827415009

4. If the sides of the nth triangle be given, the sides of the $(n-1)$th and $(n+1)$th triangles may be found as follows:

Let p_n, b_n, h_n be the sides of the nth triangle; $p_n - d$, $b_n - d$, $md - h_n$ the sides of the $(n-1)$th triangle; and $p_n + D$, $b_n + D$, $mD - h_n$ the sides of the $(n+1)$th triangle.

Then
$$\left.\begin{aligned}(p_n - d)^2 + (b_n - d)^2 &= (md - h_n)^2\\ (p_n + D)^2 + (b_n + D)^2 &= (mD - h_n)^2\end{aligned}\right\}, \quad \ldots\ldots\ldots (11)$$

from which, since $p_n + b_n = h_n$,

$$\left.\begin{aligned}d &= \frac{2mh_n - 2(b_n + p_n)}{m^2 - 2}\\ D &= \frac{2mh_n + 2(b_n + p_n)}{m^2 - 2}\end{aligned}\right\}, \quad \ldots\ldots\ldots (12)$$

Take $m = 2$, then
$$\left.\begin{aligned}d &= 2h_n - (b_n + p_n)\\ D &= 2h_n + (b_n + p_n)\end{aligned}\right\}, \quad \ldots\ldots\ldots (13)$$

$$\left.\begin{aligned}p_{n-1} &= b_n + 2p_n - 2h_n\\ b_{n-1} &= 2b_n + p_n - 2h_n\\ h_{n-1} &= 3h_n - 2b_n - 2p_n\end{aligned}\right\}, \quad \ldots\ldots\ldots (14)$$

and
$$\left.\begin{aligned}p_{n+1} &= 2h_n + b_n + 2p_n\\ b_{n+1} &= 2h_n + 2b_n + p_n\\ h_{n+1} &= 3h_n + 2b_n + 2p_n\end{aligned}\right\} \quad \ldots\ldots\ldots (15)$$

We have also the relations

$$\left.\begin{aligned}p_n &= 6p_{n-1} - p_{n-2} + 2\\ b_n &= 6b_{n-1} - b_{n-2} - 2\\ h_n &= 6h_{n-1} - h_{n-2}\end{aligned}\right\} \quad \ldots\ldots\ldots (19)$$

Hence, by the rules of recurring series, the sides of the nth triangle are the coefficients of r^n, s^n, t^n in the expansion of

$$\frac{3r-r^2}{(1-r)(1-6r+r^2)},\ \frac{4s-7s^2+s^3}{(1-s)(1-6s+s^2)} \text{ and } \frac{5t-t^2}{1-6t+t^2}$$

respectively.

SOLUTION OF THE GENERAL BIQUADRATIC EQUATION.

BY NEWTON FITZ, NORFOLK, VA.

ASSUME the identity
$x^4+ax^3+bx^2+cx+d=[x^2+(\frac{1}{2}a+\sqrt{q})x+p+r\sqrt{q}]\,[x^2+(\frac{1}{2}a-\sqrt{q})x+p-r\sqrt{q}]$. By multiplication we obtain
$x^4+ax^3+bx^2+cx+d=x^4+ax^3+(2p+\frac{1}{4}a^2-q)x^2+(ap-2rq)x+p^2-r^2q$,
whence $b=2p+\frac{1}{4}a^2-q$, $\quad c=ap-2rq$, $\quad d=p^2-r^2q$.

Eliminating r, and q, we have

$$p^3-\tfrac{1}{2}bp^2+\tfrac{1}{4}(ac-4d)p+\tfrac{1}{8}(4bd-a^2d-c^2)=0.$$

Either root of this cubic may be used in the following expressions for the roots of the biquadratic

$$\begin{aligned}
x' &= -\tfrac{1}{4}a+\tfrac{1}{2}\sqrt{q}+\tfrac{1}{2}\sqrt{[\tfrac{1}{4}a^2+q-4p+(a-4r)\sqrt{q}]},\\
x'' &= -\tfrac{1}{4}a+\tfrac{1}{2}\sqrt{q}-\tfrac{1}{2}\sqrt{[\tfrac{1}{4}a^2+q-4p+(a-4r)\sqrt{q}]},\\
x''' &= -\tfrac{1}{4}a-\tfrac{1}{2}\sqrt{q}+\tfrac{1}{2}\sqrt{[\tfrac{1}{4}a^2+q-4p-(a-4r)\sqrt{q}]},\\
x'''' &= -\tfrac{1}{4}a-\tfrac{1}{2}\sqrt{q}-\tfrac{1}{2}\sqrt{[\tfrac{1}{4}a^2+q-4p-(a-4r)\sqrt{q}]}.
\end{aligned}$$

Substituting for r and q, their values in terms of p, a, b, c, and d, gives

$$x=-\frac{a}{4}\pm\frac{1}{4}\sqrt{\left(8p+a^2-4b\right)}\pm\frac{1}{2}\sqrt{\left(\frac{a^2}{2}-2p-b\pm\frac{4ap+a^3-4ab+4c}{\sqrt{(8p+a^2-4b)}}\right)}$$

As p has, in general, three different values, there will be twelve different expressions for the roots of the biquadratic but only four different values, consequently there must exist relations of identity and of symmetry as follows: p_1, p_2, p_3 being the roots of the cubic, and f_1, f_2, f_3, f_4 representing the four forms of the roots of the biquadratic,

$$x'=f_1p_1=f_3p_2=f_4p_3, \qquad x''=f_2p_1=f_4p_2=f_3p_3,$$
$$x'''=f_3p_1=f_1p_2=f_2p_3, \qquad x''''=f_4p_1=f_2p_2=f_1p_3.$$

The general expression for the root of an equation of the nth degree will contain n variables, and by substituting for each of these n variables conjugate quadratic surds having the same radical part we obtain the expression for the roots of an equation whose degree is $2n$, as for example,